PARAMETERIZED TELESCOPING PROVES ALGEBRAIC INDEPENDENCE OF SUMS

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Abstract. Usually creative telescoping is used to derive recurrences for sums. In this article we show that the non-existence of a creative telescoping solution, and more generally, of a parameterized telescoping solution, proves algebraic independence of certain types of sums. Combining this fact with summation-theory shows transcendence of whole classes of sums. Moreover, this result throws new light on the question why, e.g., Zeilberger's algorithm fails to find a recurrence with minimal order.

1. Introduction

Telescoping [Gos78] and creative telescoping [Zei91, PWZ96] for hypergeometric terms and its variations [PS95, Pau95, PR97, BP99] are standard tools in symbolic summation. All these techniques are covered by the following formulation of the parameterized telescoping problem: **Given** sequences $f_1(k),\dots,f_d(k)$ over a certain field $\mathbb{K}$, **find**, if possible, constants $c_1,\dots,c_d\in\mathbb{K}$ and a sequence $g(k)$ such that

$$g(k+1)-g(k)=c_1f_1(k)+\cdots+c_df_d(k). \tag{1.1}$$

If one succeeds in this task, one gets, with some mild extra conditions, the sum-relation

$$g(n+1)-g(r)=c_1\sum_{k=r}^{n}f_1(k)+\cdots+c_d\sum_{k=r}^{n}f_d(k) \tag{1.2}$$

for some $r\in\mathbb{N}=\{0,1,\dots\}$ big enough. Note that $d=1$ gives telescoping. Moreover, given a bivariate sequence $f(m,k)$, one can set $f_i(k):=f(m+i-1,k)$ which corresponds to creative telescoping.

Since Karr's summation algorithm [Kar81] and its extensions [Sch05c, Sch08] can solve the parameterized telescoping problem in the difference field setting of $\Pi\Sigma^*$-fields, we get a rather flexible algorithm which is implemented in the package Sigma [Sch04b, Sch07b]: the $f_i(k)$ can be arbitrarily nested sums and products.

In this article we apply $\Pi\Sigma^*$-field theory [Kar81, Sch01] to get new theoretical insight: If there is no solution to (1.1) within a given $\Pi\Sigma^*$-field setting, then the sums in (1.2) can be represented in a larger $\Pi\Sigma^*$-field by transcendental extensions; see Theorem 3.1. Motivated by this fact, we construct a difference ring monomorphism which links elements from the larger $\Pi\Sigma^*$-field to the sums

$$S_1(n)=\sum_{k=r}^{n}f_1(k),\dots,S_d(n)=\sum_{k=r}^{n}f_d(k) \tag{1.3}$$

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in the ring of sequences over $\mathbb{K}$. In particular, this construction transfers the transcendence properties from the $\Pi\Sigma^*$-world into the sequence domain. In order to accomplish this task, we restrict to generalized d'Alembertian extensions, a subclass of $\Pi\Sigma^*$-extensions, which cover all those sum-product expressions that occurred in practical problem solving so far.

Summarizing, parameterized telescoping in combination with $\Pi\Sigma^*$-fields gives a criterion to check algorithmically the transcendence of sums of type (1.3); see Theorem 5.1. Combining this criterion with results from summation theory, like [Abr71, Pau95, Abr03, Sch07a], shows that whole classes of sequences are transcendental. E.g., the harmonic numbers $\{H_n^{(i)}\,|\,i\geq 1\}$ with $H_n^{(i)}:=\sum_{k=1}^n\frac{1}{k^i}$ are algebraically independent over $\mathbb{Q}(n)$.

Moreover, we derive new insight for which sums creative telescoping, in particular Zeilberger's algorithm, finds the optimal recurrence and for which input classes it might fail to compute a recurrence with minimal order.

The general structure of this article is as follows. In Section 2 we present the basic notions of difference fields, and we introduce $\Pi\Sigma^*$-extensions together with the subclass of generalized d'Alembertian extensions. In Section 3 we show the correspondence of parameterized telescoping and the construction of a certain type of Σ^*-extensions. In Section 4 we construct a difference ring monomorphism that carries over the transcendence properties from a given generalized d'Alembertian extension to the ring of sequences. This leads to a transcendence decision criterion of sequences in terms of generalized d'Alembertian extensions in Section 5. In Sections 6–8 we apply our criterion to the rational case, hypergeometric case, and to nested sums. Finally, we present the analogous criterion for products in Section 9.

2. Basic notions: $\Pi\Sigma^*$-extensions and generalized d'Alembertian extension

Subsequently, we introduce the basic concepts of difference fields that shall pop up later.

A *difference ring*[1] (resp. field) $(\mathbb{A},\sigma)$ is a ring $\mathbb{A}$ (resp. field) with a ring automorphism (resp. field automorphism) $\sigma:\mathbb{A}\to\mathbb{A}$. The set of constants $\text{const}_\sigma\mathbb{A}=\{k\in\mathbb{A}\,|\,\sigma(k)=k\}$ forms a subring (resp. subfield) of $\mathbb{A}$. In this article we always assume that $\text{const}_\sigma\mathbb{A}$ is a field, which we usually denote by $\mathbb{K}$. We call $\text{const}_\sigma\mathbb{A}$ the *constant field* of $(\mathbb{A},\sigma)$.

A *difference ring homomorphism* (resp. difference ring monomorphism) $\tau:\mathbb{A}_1\to\mathbb{A}_2$ between two difference rings $(\mathbb{A}_1,\sigma_1)$ and $(\mathbb{A}_2,\sigma_2)$ is a ring homomorphism (resp. ring monomorphism) with the additional property that $\tau(\sigma_1(f))=\sigma_2(\tau(f))$ for all $f\in\mathbb{A}_1$.

A difference ring (resp. difference field) $(\mathbb{E},\sigma)$ is a *difference ring extension* (resp. *difference field extension*) of a difference ring (resp. difference field) $(\mathbb{A},\sigma')$ if $\mathbb{A}$ is a subring (resp. subfield) of $\mathbb{E}$ and $\sigma'(f)=\sigma(f)$ for all $f\in\mathbb{A}$; since σ and σ' agree on $\mathbb{A}$, we do not distinguish them anymore.

Now we are ready to define $\Pi\Sigma^*$-extensions and generalized d'Alembertian extensions in which we will represent our indefinite nested sums and products.

Definition 2.1. A difference field extension $(\mathbb{F}(t),\sigma)$ of $(\mathbb{F},\sigma)$ is called a $\Pi\Sigma^*$-*extension* if both difference fields share the same field of constants, t is transcendental over $\mathbb{F}$, and $\sigma(t)=t+a$ for some $a\in\mathbb{F}^*$ (a sum) or $\sigma(t)=a\,t$ for some $a\in\mathbb{F}^*$ (a product). If $\sigma(t)/t\in\mathbb{F}$ (resp. $\sigma(t)-t\in\mathbb{F}$), we call the extension also a Π-*extension* (resp. Σ^*-*extension*). In short, we say that $(\mathbb{F}(t_1)\dots(t_e),\sigma)$ is a $\Pi\Sigma^*$-extension (resp. Π-extension, Σ^*-extension) of $(\mathbb{F},\sigma)$ if the extension is given by a tower of $\Pi\Sigma^*$-extensions (resp. Π-extensions, Σ^*-extensions). A $\Pi\Sigma^*$-*field* $(\mathbb{K}(t_1)\dots(t_e),\sigma)$ over $\mathbb{K}$ is a $\Pi\Sigma^*$-extension of $(\mathbb{K},\sigma)$ with constant field $\mathbb{K}$.

[1]All fields and rings are of characteristic 0 and commutative

Example 2.2. Consider the difference field $(\mathbb{Q}(m)(k)(b)(h),\sigma)$ with $\sigma(k)=k+1$, $\sigma(b)=\frac{m-k}{k+1}b$, $\sigma(h)=h+\frac{1}{k+1}$, and $\mathrm{const}_\sigma\mathbb{Q}(m)(k)(b)(h)=\mathbb{Q}(m)$. The extensions k, b, and h form $\Pi\Sigma^*$-extensions over the fields below. $(\mathbb{Q}(m)(k)(b)(h),\sigma)$ is a $\Pi\Sigma^*$-field over $\mathbb{Q}(m)$.

The following theorem tells us how one can check if an extension is a $\Pi\Sigma^*$-extension.

Theorem 2.3 ([Kar81]). *Let $(\mathbb{F}(t),\sigma)$ be a difference field extension of $(\mathbb{F},\sigma)$ with $\sigma(t)=\alpha\,t+\beta$ where $\alpha\in\mathbb{F}^*$ and $\beta\in\mathbb{F}$. Then:*
(1) This is a Σ^*-extension iff $\alpha=1$ and there is no $g\in\mathbb{F}$ such that $\sigma(g)-g=\beta$.
(2) This is a Π-extension iff $t\neq 0$, $\beta=0$ and there are no $n\neq 0$, $g\in\mathbb{F}^*$ such that $\sigma(g)=\alpha^n g$.

The following remarks are in place:

(1) If $(\mathbb{F},\sigma)$ is a $\Pi\Sigma^*$-field, algorithms are available which make Theorem 2.3 completely constructive; see [Kar81, Sch05c].
(2) We emphasize that we have a first criterion for transcendence in a difference field: if there is no telescoping solution, then we can adjoin the sum as a transcendental extension without extending the constant field. This criterion will be generalized to parameterized telescoping; see Theorem 3.1. For the product case see Theorem 9.1.

Theorem 2.4 states how solutions g of $\sigma(g)-g=f$ or $\sigma(g)=f\,g$ look like in certain types of extensions. The first part follows by [Kar81, Sec. 4.1] and the second part follows by [Sch05b, Lemma 6.8]. These results are crucial ingredients to prove Theorems 3.1 and 9.1.

Theorem 2.4. *Let $(\mathbb{F}(t_1,\dots,t_d),\sigma)$ be a $\Pi\Sigma^*$-extension of $(\mathbb{F},\sigma)$ with constant field $\mathbb{K}$ such that for all $1\leq i\leq d$ we have $\sigma(t_i)=\alpha_i t_i+\beta_i$ with $\alpha_i,\beta_i\in\mathbb{F}$. Let $f\in\mathbb{F}$ and $g\in\mathbb{F}(t_1,\dots,t_d)$.*

(1) *If $\sigma(g)-g=f$, then $g=\sum_{i=1}^{d}c_i\,t_i+w$ with $w\in\mathbb{F}$, $c_i\in\mathbb{K}$; if $\alpha_i\neq 1$, then $c_i=0$.*
(2) *If $\sigma(g)=f\,g$, then $g=w\prod_{i=1}^{d}t_i^{c_i}$ with $w\in\mathbb{F}$ and $c_i\in\mathbb{Z}$; if $\beta_i\neq 0$, then $c_i=0$.*

Subsequently, we will restrict to the following type of extensions.

Definition 2.5. We call $\Pi\Sigma^*$-extension $(\mathbb{F}(t_1)\dots(t_e),\sigma)$ of $(\mathbb{F},\sigma)$ with $\sigma(t_i)=\alpha_i\,t_i+\beta_i$ *generalized d'Alembertian extension* if $\alpha_i\in\mathbb{F}$ and $\beta_i\in\mathbb{F}[t_1,\dots,t_{i-1}]$ for all $1\leq i\leq e$.

Remark 2.6. Subsequently, we exploit the following fact: One can reorder a generalized d'Alembertian extension to $\mathbb{F}(p_1)\dots(p_u)(s_1)\dots(s_v)$ with $u,v\geq 0$ where $\frac{\sigma(p_i)}{p_i}\in\mathbb{F}$ for $1\leq i\leq u$ and $\sigma(s_i)-s_i\in\mathbb{F}[p_1,\dots,p_u,s_1,\dots,s_{i-1}]$ for $1\leq i\leq v$.

It is easy to see that $(\mathbb{F}[t_1,\dots,t_e],\sigma)$ is a difference ring extension of $(\mathbb{F},\sigma)$. Moreover, if $f\in\mathbb{F}[t_1,\dots,t_e]$, then there are no solutions in $\mathbb{F}(t_1,\dots,t_e)\setminus\mathbb{F}[t_1,\dots,t_e]$.

Theorem 2.7. *Let $(\mathbb{F}(t_1)\dots(t_e),\sigma)$ be a generalized d'Alembertian extension of $(\mathbb{F},\sigma)$ and suppose that $g\in\mathbb{F}(t_1)\dots(t_e)$. Then $\sigma(g)-g\in\mathbb{F}[t_1,\dots,t_e]$ if and only if $g\in\mathbb{F}[t_1,\dots,t_e]$.*

Proof. The direction from left to right is clear by the definition of generalized d'Alembertian extensions. We prove the other direction by induction on the number of extensions. For $e=0$ nothing has to be shown. Now suppose that the theorem holds for e extensions and consider the generalized d'Alembertian extension $(\mathbb{F}(t_1)\dots(t_{e+1}),\sigma)$ of $(\mathbb{F},\sigma)$. By Remark 2.6 we can bring $\mathbb{F}(t_1)\dots(t_{e+1})$ to a form where all Σ^*-extensions are on top. Write $t:=t_{e+1}$ and let $\sigma(t)=\alpha\,t+\beta$ with $\alpha\in\mathbb{F}^*$ and $\beta\in\mathbb{F}[t_1,\dots,t_e]$. Now suppose that $\sigma(g)-g=f$ where $g\in\mathbb{F}(t_1,\dots,t_e,t)\setminus\mathbb{F}[t_1,\dots,t_e,t]$ and $f\in\mathbb{F}[t_1,\dots,t_e][t]$. Note that $g\in\mathbb{F}(t_1,\dots,t_e)[t]$; see, e.g., [Sch07a, Lemma 3.1]. Hence we can write $g=\sum_{i=0}^{d}g_it^i$ with $g_i\in\mathbb{F}(t_1,\dots,t_e)$. If $e=0$, we are done. Otherwise, suppose that $e>0$ and let $j\geq 0$ be maximal such

that $g_j \in \mathbb{F}(t_1,\dots,t_e)\setminus\mathbb{F}[t_1,\dots,t_e]$. Define $g' := \sum_{i=0}^{j} g_i t^i \in \mathbb{F}(t_1,\dots,t_e)[t]$ and $f' := f - \big(\sigma(\sum_{i=j+1}^{d} g_i t^i) - \sum_{i=j+1}^{d} g_i t^i\big) \in \mathbb{F}[t_1,\dots,t_e][t]$. Since $\sigma(g') - g' = f'$, $\deg(f') \leq \deg(g') = j$. By coefficient comparison we have

$$\alpha^j \sigma(g_j) - g_j = \phi \in \mathbb{F}[t_1,\dots,t_e] \tag{2.1}$$

where ϕ is the jth coefficient in f'. If $\alpha = 1$ or $j = 0$, we can apply the induction assumption and conclude that $g_j \in \mathbb{F}[t_1,\dots,t_e]$, a contradiction. Otherwise, suppose that $\alpha \neq 1$ and $j \geq 1$. Then by the assumption that all Π-extensions come first, it follows that $\sigma(t_i)/t_i \in \mathbb{F}$ for all $1 \leq i \leq e$. Reorder $\mathbb{F}(t_1,\dots,t_e)$ such that $g_j \notin \mathbb{F}(t_1,\dots,t_{e-1})[t_e]$. By Bronstein [Bro00, Cor. 3], see also [Sch04a, Cor. 1], we get $g_j = \frac{p}{t_e^m}$ for some $m > 0$ and[2] $p \in \mathbb{F}(t_1,\dots,t_{e-1})[t_e]^*$ with $t_e \nmid p$. Hence $\alpha^j \sigma(\frac{p}{t_e^m}) - \frac{p}{t_e^m} = \frac{\alpha^j \sigma(p) - a^m p}{a^m t_e^m} = \phi$ with $a := \frac{\sigma(t_e)}{t_e} \in \mathbb{F}^*$. Since $t_e \nmid p$, also $a\,t_e = \sigma(t_e) \nmid \sigma(p)$, and thus $t_e \nmid \sigma(p)$. Since (2.1) and $m > 0$, it follows that $\alpha^j \sigma(p) - a^m p = 0$, and hence $\sigma(\frac{t_e^m}{p}) = \alpha^j\, \frac{t_e^m}{p}$; a contradiction to Theorem 2.3.2 and the fact that $(\mathbb{F}(t_1,\dots,t_e)(t),\sigma)$ is a Π-extension of $(\mathbb{F}(t_1,\dots,t_e),\sigma)$. $\square$

Given a rational function field $\mathbb{F}(t)$, we say that $\frac{p}{q} \in \mathbb{F}(t)$ is in reduced representation, if $p, q \in \mathbb{F}[t]$, $\gcd(p,q) = 1$, and q is monic. The summation criterion from [Abr71],[Pau95, Prop. 3.3] and its generalization to $\Pi\Sigma^*$-extensions are substantial:

Theorem 2.8. *([Sch07a, Cor. 5.1]) Let $(\mathbb{F}(t),\sigma)$ be a $\Pi\Sigma^*$-extension of $(\mathbb{F},\sigma)$ and $\frac{p}{q} \in \mathbb{F}(t)$ be in reduced representation with $\deg(q) > 0$ and with the property that either $t \nmid q$ or $\frac{\sigma(t)}{t} \notin \mathbb{F}$. If $\gcd(\sigma^m(q), q) = 1$ for all $m > 0$, then there is no $g \in \mathbb{F}(t)$ with $\sigma(g) - g = \frac{p}{q}$.*

Corollary 2.9 is immediate; for a more general version see [Sch01, Prop. 4.1.1].

Corollary 2.9. *Let $(\mathbb{F}(t),\sigma)$ be a Π-extension of $(\mathbb{F},\sigma)$ with $\sigma(t) = \alpha\, t$, and let $w \in \mathbb{F}$ and $g \in \mathbb{F}(t)$. Then $\sigma(g) - g = w\,t$, iff $g = v\,t + c$ for $v \in \mathbb{F}$, $c \in \mathrm{const}_\sigma\mathbb{F}$ where $\alpha\,\sigma(v) - v = w$.*

Proof. Suppose that $g \in \mathbb{F}(t)$ with $\sigma(g) - g = w\,t$. By Theorem 2.8 $g \in \mathbb{F}[t]$. Moreover, by [Kar81, Cor. 2] or [Sch05a, Cor. 3], $g = v\,t + u$ with $v, u \in \mathbb{F}$. Plugging g into $\sigma(g) - g = w\,t$ and doing coefficient comparison shows that $u \in \mathrm{const}_\sigma\mathbb{F}$ and $\alpha\,\sigma(v) - v = w$. The other direction follows immediately. $\square$

3. Parameterized telescoping, $\Pi\Sigma^*$-extensions and the ring of sequences

We get the following criterion to check transcendence in a given difference field $(\mathbb{F},\sigma)$.

Theorem 3.1. *Let $(\mathbb{F},\sigma)$ be a difference field with constant field $\mathbb{K}$ and $(f_1,\dots,f_d) \in \mathbb{F}^d$. The following statements are equivalent.*

1. *There do not exist a $\mathbf{0} \neq (c_1,\dots,c_d) \in \mathbb{K}^d$ and a $g \in \mathbb{F}$ with*
$$\sigma(g) - g = c_1 f_1 + \dots + c_d f_d. \tag{3.1}$$
2. *There is a Σ^*-extension $(\mathbb{F}(t_1)\dots(t_d),\sigma)$ of $(\mathbb{F},\sigma)$ with $\sigma(t_i) = t_i + f_i$ for $1 \leq i \leq d$.*

Proof. Suppose that (3.1) holds for some $\mathbf{0} \neq (c_1,\dots,c_d) \in \mathbb{K}^d$ and $g \in \mathbb{F}$. In addition, assume that there exists a Σ^*-extension $(\mathbb{F}(t_1,\dots,t_d),\sigma)$ of $(\mathbb{F},\sigma)$ with $\sigma(t_i) = t_i + f_i$. Then $\sigma(g) - g = \sum_{i=1}^{d} c_i\big(\sigma(t_i) - t_i\big) = \sigma(\sum_{i=1}^{d} c_i\,t_i) - \sum_{i=1}^{d} c_i\,t_i$, and thus $\sigma(\sum_{i=1}^{d} c_i\,t_i - g) = \sum_{i=1}^{d} c_i\,t_i - g$. Since $\mathrm{const}_\sigma\mathbb{F}(t_1,\dots,t_d) = \mathbb{K}$, there is a $k \in \mathbb{K}$ with $\sum_{i=1}^{d} c_i\,t_i - g + k = 0$. Thus there are

[2] For a ring $\mathbb{A}$ we define $\mathbb{A}^* := \mathbb{A}\setminus\{0\}$.

algebraic relations in the t_i, a contradiction to the definition of $\Pi\Sigma^*$-extensions.
Contrary, let $i \geq 1$ be maximal such that $(\mathbb{F}(t_1, \dots, t_i), \sigma)$ is a Σ^*-extension of $(\mathbb{F}, \sigma)$; suppose that $i < d$. Then there is a $g \in \mathbb{F}(t_1, \dots, t_i)$ with $\sigma(g) - g = f_{i+1}$. By Theorem 2.4.1 there are $c_j \in \mathbb{K}$, $h \in \mathbb{F}$ with $g = h + \sum_{j=1}^{i} c_j t_j$. This shows that $\sigma(h) - h = f_{i+1} - \sum_{j=1}^{i} c_j(\sigma(t_j) - t_j) = -c_1 f_1 - \dots - c_i f_i + f_{i+1}$. Hence we get a solution for (3.1) with $\mathbf{0} \neq (c_1, \dots, c_d) \in \mathbb{K}^d$. □

Let $\mathbb{K}$ be a field with characteristic zero. The set of all sequences $\mathbb{K}^{\mathbb{N}}$ with elements $(a_n)_{n=0}^{\infty} = (a_0, a_1, a_2, \dots)$, $a_i \in \mathbb{K}$, forms a commutative ring by component-wise addition and multiplication; the field $\mathbb{K}$ can be naturally embedded by identifying $k \in \mathbb{K}$ with the sequence $\boldsymbol{k} := (k, k, k, \dots)$. In order to turn the shift-operation

$$S : (a_0, a_1, a_2, \dots) \mapsto (a_1, a_2, a_3, \dots) \tag{3.2}$$

to an automorphism, we follow the construction from [PWZ96, Sec. 8.2]: We define an equivalence relation $\sim$ on $\mathbb{K}^{\mathbb{N}}$ with $(a_n)_{n=0}^{\infty} \sim (b_n)_{n=0}^{\infty}$ if there exists a $\delta \geq 0$ such that $a_k = b_k$ for all $k \geq \delta$. The equivalence classes form a ring which is denoted by $S(\mathbb{K})$; the elements of $S(\mathbb{K})$ will be denoted, as above, by sequence notation. Now it is immediate that $S : S(\mathbb{K}) \to S(\mathbb{K})$ with (3.2) forms a ring automorphism. The difference ring $(S(\mathbb{K}), S)$ is called the *ring of* $\mathbb{K}$*-sequences* or in short the *ring of sequences*.

The main construction of our article is that the polynomial ring $\mathbb{F}[t_1, \dots, t_e]$ of a generalized d'Alembertian extension $(\mathbb{F}(t_1) \dots (t_e), \sigma)$ of $(\mathbb{F}, \sigma)$ with constant field $\mathbb{K}$ can be embedded in the ring of sequences $(S(\mathbb{K}), S)$, provided that $(\mathbb{F}, \sigma)$ can be embedded in $(S(\mathbb{K}), S)$. More precisely, we will construct a difference ring monomorphism $\tau : \mathbb{F}[t_1, \dots, t_e] \to S(\mathbb{K})$ where the constants $k \in \mathbb{K}$ are mapped to $\boldsymbol{k} = (k, k, \dots)$. We will call such a difference ring homomorphism (resp. monomorphism) also a $\mathbb{K}$*-homomorphism* (resp. $\mathbb{K}$*-monomorphism*). Then the main consequence is that the transcendence properties of generalized d'Alembertian extensions, in particular Theorem 3.1, can be carried over to $S(\mathbb{K})$; see Theorem 5.1.

4. The monomorphism construction

In the following we will construct the $\mathbb{K}$-monomorphism as mentioned in the end of Section 3. Here we use the following lemma which is inspired by [NP97]; the proof is obvious.

Lemma 4.1. *Let* $(\mathbb{A}, \sigma)$ *be a difference ring with constant field* $\mathbb{K}$. *If* $\tau : \mathbb{A} \to S(\mathbb{K})$ *is a* $\mathbb{K}$*-homomorphism, there is a map* $\mathrm{ev} : \mathbb{A} \times \mathbb{N} \to \mathbb{K}$ *with*

$$\tau(f) = (\mathrm{ev}(f, 0), \mathrm{ev}(f, 1), \dots) \tag{4.1}$$

for all $f \in \mathbb{A}$ *which has the following properties: For all* $c \in \mathbb{K}$ *there is a* $\delta \geq 0$ *with*

$$\forall i \geq \delta : \ \mathrm{ev}(c, i) = c; \tag{4.2}$$

for all $f, g \in \mathbb{A}$ *there is a* $\delta \geq 0$ *with*

$$\forall i \geq \delta : \ \mathrm{ev}(f\, g, i) = \mathrm{ev}(f, i)\, \mathrm{ev}(g, i), \tag{4.3}$$

$$\forall i \geq \delta : \ \mathrm{ev}(f + g, i) = \mathrm{ev}(f, i) + \mathrm{ev}(g, i); \tag{4.4}$$

and for all $f \in \mathbb{A}$ *and* $j \in \mathbb{Z}$ *there is a* $\delta \geq 0$ *with*

$$\forall i \geq \delta \ \mathrm{ev}(\sigma^j(f), i) = \mathrm{ev}(f, i + j). \tag{4.5}$$

Conversely, if we have a map $\mathrm{ev} : \mathbb{A} \times \mathbb{N} \to \mathbb{K}$ *with* (4.2), (4.3), (4.4) *and* (4.5), *then the map* $\tau : \mathbb{A} \to S(\mathbb{K})$ *defined by* (4.1) *forms a* $\mathbb{K}$*-homomorphism.*

In order to take into account the constructive aspects, we introduce the following functions.

Definition 4.2. Let $(\mathbb{A},\sigma)$ be a difference ring and $\tau:\mathbb{A}\to S(\mathbb{K})$ be a $\mathbb{K}$-homomorphism defined by (4.1). τ is called *operation-bounded* by $L:\mathbb{A}\to\mathbb{N}$ if for all $f\in\mathbb{A}$ and $j\in\mathbb{Z}$ with $\delta=\delta(f,j):=L(f)+\max(0,-j)$ we have (4.5) and for all $f,g\in\mathbb{A}$ with $\delta=\delta(f,g):=\max(L(f),L(g))$ we have (4.3) and (4.4). Moreover, we require that for all $f\in\mathbb{A}$ and all $j\in\mathbb{Z}$ we have $L(\sigma^j(f))\leq L(f)+\max(0,-j)$; such a function is also called *o-function.* τ is called zero-bounded by $Z:\mathbb{A}\to\mathbb{N}$ if for all $f\in\mathbb{A}^*$ and all $i\geq Z(f)$ we have $\mathrm{ev}(f,i)\neq 0$; such a function is also called *z-function.*

Lemma 4.3. *Let $(\mathbb{A},\sigma)$ be a difference field with constant field $\mathbb{K}$. If $\tau:\mathbb{A}\to S(\mathbb{K})$ is a $\mathbb{K}$-homomorphism with (4.1), then for all $f\in\mathbb{A}$ we have $\mathrm{ev}(\frac{1}{f},i)=\frac{1}{\mathrm{ev}(f,i)}$ for big enough i. In particular, there is a z-function for τ.*

Proof. $\tau(f^{-1})$ is the inverse of $\tau(f)$, i.e., $\tau(\frac{1}{f})=\frac{1}{\tau(f)}$. Hence, $\mathrm{ev}(\frac{1}{f},k)=\frac{1}{\mathrm{ev}(f,k)}$ for all $k\geq\delta$ for some $\delta\geq 0$. This implies $\mathrm{ev}(f,k)\neq 0$ for all $k\geq\delta$. Hence there is a z-function. $\square$

The next lemma is the crucial tool to design step by step a $\mathbb{K}$-monomorphism for a generalized d'Alembertian extension. This construction will be used in Theorem 5.1.

Lemma 4.4. *Let $(\mathbb{F}(t_1)\dots(t_e)(t),\sigma)$ be a generalized d'Alembertian extension of $(\mathbb{F},\sigma)$ with $\mathbb{K}:=\mathrm{const}_\sigma\mathbb{F}$ and $\sigma(t)=\alpha\,t+\beta$. Let $\tau:\mathbb{F}[t_1]\dots[t_e]\to S(\mathbb{K})$ be a $\mathbb{K}$-homomorphism with (4.1) together with an o-function L. Then:*

(1) There is a $\mathbb{K}$-homomorphism $\tau':\mathbb{F}[t_1]\dots[t_e][t]\to S(\mathbb{K})$ with $\tau'(f)=(\mathrm{ev}'(f,l))_{l\geq 0}$ for all $f\in\mathbb{F}[t_1,\dots,t_e][t]$ such that $\tau'(f)=\tau(f)$ for all $f\in\mathbb{F}[t_1,\dots,t_e]$; if $\beta=0$, $\mathrm{ev}'(t,k)\neq 0$ for all $k\geq r$ for some $r\in\mathbb{N}$. Such a τ' is uniquely determined by

$$\mathrm{ev}'(t,k)=\begin{cases} c\displaystyle\prod_{i=r}^{k}\mathrm{ev}(\alpha,i-1) & \text{if } \sigma(t)=\alpha\,t\\ \displaystyle\sum_{i=r}^{k}\mathrm{ev}(\beta,i-1)+c & \text{if } \sigma(t)=t+\beta,\end{cases} \tag{4.6}$$

up to the choice of $r\in\mathbb{N}$ and $c\in\mathbb{K}$; we require $c\neq 0$, if $\beta=0$.
(2) If τ is injective, τ' is injective.
(3) There is an o-function for τ'.
(4) If there is a computable z-function for τ restricted on $\mathbb{F}$ and a computable o-function for τ, then there is a computable o-function for τ'; r in (4.6) can be computed.

Proof. **(1)** By Lemma 4.3 there is a z-function $Z:\mathbb{F}\to\mathbb{N}$ for τ restricted on $\mathbb{F}$. Let τ be defined by (4.1), denote $\mathbb{A}:=\mathbb{F}[t_1,\dots,t_e]$ and let $\sigma(t)=\alpha\,t+\beta$ with $\alpha\in\mathbb{F}^*$ and $\beta\in\mathbb{A}$.
Let $\mathrm{ev}'(t,k)$ be defined as in (4.6) (r will be specified later for the concrete cases $\alpha=1$ or $\beta=0$). We extend ev from $\mathbb{A}$ to $\mathbb{A}[t]$ by

$$\mathrm{ev}'(\sum_{i=0}^{n}f_it^i,k)=\sum_{i=0}^{n}\mathrm{ev}(f_i,k)\mathrm{ev}'(t,k)^i.$$

First suppose that $\alpha = 1$. Let $r := L(f) + 1$ and consider the sequence given by (4.6) for some $c \in \mathbb{K}$. Let $j \geq 0$. Then by construction and by the choice of r we have for all $k \geq r$:

$$\begin{aligned}\mathrm{ev}'(\sigma^j(t), k) &= \mathrm{ev}'(t + \sum_{i=0}^{j-1}\sigma^i(\beta), k) = \mathrm{ev}'(t, k) + \sum_{i=0}^{j-1}\mathrm{ev}(\sigma^i(\beta), k) + c \\ &= \mathrm{ev}'(t, k) + \sum_{i=0}^{j-1}\mathrm{ev}(\beta, k+i) + c = \mathrm{ev}'(t, k+j) = \mathrm{ev}'(t, k+j).\end{aligned}$$

Similarly, if $j < 0$, then $\mathrm{ev}'(t, k+j) = \mathrm{ev}'(\sigma^j(t), k)$ for all $k \geq r - j$. This proves (4.5) for $f = t$ and all $j \in \mathbb{Z}$ with $\delta = r + \max(-j, 0)$ (ev is replaced by ev′). Now suppose that $\beta = 0$. Let $r := \max(Z(\alpha), L(f)) + 1$ and $c \in \mathbb{K}^*$, and consider the sequence given by (4.6). Analogously, it follows that for all $j \in \mathbb{Z}$ we have $\mathrm{ev}'(t, k+j) = \mathrm{ev}'(\sigma^j(t), k)$ for all $k \geq r + \max(-j, 0)$. Moreover, since $\mathrm{ev}(\alpha, i-1) \neq 0$ for all $i \geq r$, $\mathrm{ev}'(t, k) \neq 0$ for all $k \geq r$. Moreover, if we choose $\delta \geq r$ big enough (depending on the f_i), we get (4.5) for $f = \sum_{i=0}^{n} f_i t^i$. Similarly, we can find for all $f, g \in \mathbb{A}[t]$ a $\delta \geq 0$ with (4.3) and (4.4). Moreover, (4.2) holds, since ev′ restricted on $\mathbb{A}$ equals ev. Summarizing, if we define $\tau' : \mathbb{A}[t] \to S(\mathbb{K})$ by $\tau'(f) = (\mathrm{ev}'(f, i))_{i\geq 0}$ for all $f \in \mathbb{A}[t]$, τ' forms a $\mathbb{K}$-homomorphism by Lemma 4.1. Note: if $\beta = 0$, then $\mathrm{ev}'(t, k) \neq 0$ for all $k \geq r$.

Furthermore, this construction is unique up to c and r in (4.6). Namely, take any other τ_2 with $\tau_2(f) = \tau(f)$ for $f \in \mathbb{F}[t_1, \dots, t_e]$ and define $T := \tau_2(t)$; if $\beta = 0$, we require in addition that T is nonzero from a certain point on. Then $S(T) = S(\tau_2(t)) = \tau_2(\sigma(t)) = \tau_2(\alpha\, t + \beta) = \tau_2(\alpha)T + \tau(\beta)$. Note that $\tau_2(1) = \mathbf{1}$ and $\tau_2(0) = \mathbf{0}$. Hence, if $\alpha = 1$, $S(T) = T + \tau(\beta)$, and therefore $S(T - \tau(t)) = T - \tau(t)$, i.e., $T = \tau(t) + \boldsymbol{d}$ for some constant $d \in \mathbb{K}$. Similarly, if $\beta = 0$, $S(T) = \tau(\alpha)T$. Since $\tau'(t)$ is non-zero from the point r on, one can take the inverse $1/\tau'(t)$ and gets $S(\frac{T}{\tau'(t)}) = \frac{T}{\tau'(t)}$. Hence $\frac{T}{\tau'(t)} = \boldsymbol{d}$ with $d \in \mathbb{K}$, i.e., $T = \boldsymbol{d}\tau'(t)$. Since T is nonzero for almost all entries, $d \neq 0$. This shows that τ_2 can be defined by (4.6) up to a constant $d \in \mathbb{K}$; $d \neq 0$, if $\beta = 0$. Note: a different r can be compensated by an appropriate choice of d.

(2) Suppose that τ is a $\mathbb{K}$-monomorphism, but the extended $\mathbb{K}$-homomorphism τ' is not injective. Then take $f = \sum_{i=0}^{n} f_i t^i \in \mathbb{A}[t]^*$ with $\tau'(f) = \mathbf{0}$ where $\deg(f) = n$ is minimal. Note that $f \notin \mathbb{A}$, otherwise $\tau'(f) = \tau(f) = \mathbf{0}$; a contradiction that τ is injective. Hence, $n > 0$. Define

$$h := \sigma(f_n)\alpha^n f - f_n\sigma(f) = \sigma(f_n)\alpha^n\sum_{i=0}^{n} f_i t^i - f_n\sum_{i=0}^{n}\sigma(f_i)(\alpha\, t + \beta)^i \in \mathbb{A}[t]. \tag{4.7}$$

Since $\mathbf{0} = S(\mathbf{0}) = S(\tau'(f)) = \tau'(\sigma(f))$, we have $\tau'(h) = \tau(\sigma(f_n)\alpha^n)\tau'(f) - \tau(f_n)\tau'(\sigma(f)) = \mathbf{0}$. In addition, $\deg(h) < n$ by construction. It follows that $h = 0$, otherwise we get a contradiction to the minimality of n. With (4.7) we get $\sigma(f)/f \in \mathbb{F}(t_1)\dots(t_e)$ with $f \notin \mathbb{F}(t_1)\dots(t_e)$. If t is a Σ^*-extension, we get a contradiction by Theorem 2.4.1. Otherwise, suppose that t is a Π-extension. W.l.o.g. suppose that all Π-extensions come first, say $t_1, \dots, t_r$ ($r \geq 0$). Then $f = u\, t_1^{m_1} \dots t_r^{m_r} t^n$ with $m_1 \dots, m_r \in \mathbb{N}$ and $u \in \mathbb{F}^*$ by Theorem 2.4.2. Hence $\mathbf{0} = \tau'(f) = \tau(u)\, \tau(t_1)^{m_1} \dots \tau(t_r)^{m_r} \tau'(t)^n$. Note that for $1 \leq i \leq r$, $\tau(t_i)$ is non-zero from a point on (for all $r \geq 0$, $S^r(\tau(t_i)) = \boldsymbol{v}\tau(t_i)$ for some $\boldsymbol{v} \in S(\mathbb{K})$; hence if infinitely many zeros occur, we can variate r to prove $\tau(t_i) = \mathbf{0}$; a contradiction). Since also $\tau(u) \neq \mathbf{0}$ (τ is injective and $u \neq 0$), $\tau'(t)$ has infinitely many zeros, a contradiction to our construction of τ'. Summarizing, τ' is injective.

(3) Note that the r for (4.6) can be defined by $r := \max(Z(\alpha), L(\alpha)) + 1$ or $r := L(\beta) + 1$,

respectively. Define $L' : \mathbb{A}[t] \to \mathbb{N}$ by

$$L'(f) = \begin{cases} L(f) & \text{if } f \in \mathbb{A}, \\ \max(r, L(f_1), \dots, L(f_n)) & \text{if } f = \sum_{i=0}^{n} f_i t^i \notin \mathbb{A}. \end{cases}$$

Then one can check that L' is an o-function for τ'. E.g., for $f = \sum_{i=0}^{m} f_i t^i$, $g = \sum_{i=0}^{n} g_i t^i$:

$$\begin{aligned} \mathrm{ev}'(f\,g, k) &= \mathrm{ev}'(\sum_{j=0}^{m+n} t^j \sum_{i=0}^{j} f_i g_{j-i}, k) = \sum_{j=0}^{m+n} \mathrm{ev}'(t,k)^j \sum_{i=0}^{j} \mathrm{ev}(f_i g_{j-i}, k) \\ &= \Big(\sum_{i=0}^{m} \mathrm{ev}(f_i, k)\mathrm{ev}'(t,k)^i\Big)\Big(\sum_{j=0}^{n} \mathrm{ev}(g_j, k)\mathrm{ev}'(t,k)^j\Big) = \mathrm{ev}(f,k)\mathrm{ev}(g,k) \end{aligned}$$

for all $k \geq \max(r, L(f_0), \dots, L(f_m), L(g_0), \dots, L(g_n)) = \max(L'(f), L'(g))$.
(4) In particular, if L and Z are computable, then L' is computable; by construction an appropriate r in (4.6) can be computed. □

By iterative application of the previous lemma we arrive at

Theorem 4.5. *Let $(\mathbb{F}(t_1)\dots(t_e), \sigma)$ be a generalized d'Alembertian-extension of $(\mathbb{F}, \sigma)$ with $\mathbb{K} := \mathrm{const}_\sigma \mathbb{F}$. If there is a $\mathbb{K}$-homomorphism/$\mathbb{K}$-monomorphism $\tau : \mathbb{F} \to S(\mathbb{K})$ with an o-function L, then there is a $\mathbb{K}$-homomorphism/$\mathbb{K}$-monomorphism $\tau' : \mathbb{F}[t_1, \dots, t_e] \to S(\mathbb{K})$ with $\tau'(f) = \tau(f)$ for all $f \in \mathbb{F}$ together with an o-function L' for τ'. If L is computable and τ has a computable z-function, L' is computable.*

In order to apply Theorem 4.5, we must be able to embed the ground field $(\mathbb{F}, \sigma)$ with $\mathbb{K} = \mathrm{const}_\sigma \mathbb{F}$ in the ring of sequences $(S(\mathbb{K}), S)$ by a $\mathbb{K}$-monomorphism. We give a criterion when this is possible in Theorem 4.6. Applying this result we get, e.g., $\mathbb{K}$-monomorphisms for the rational case, the q-rational case and the mixed case.

Theorem 4.6. *Let $(\mathbb{F}(t), \sigma)$ be a $\Pi\Sigma^*$-extension of $(\mathbb{F}, \sigma)$ with constant field $\mathbb{K}$ and let $\tau : \mathbb{F}[t] \to S(\mathbb{K})$ be a $\mathbb{K}$-homomorphism (resp. $\mathbb{K}$-monomorphism).*

1. *There is a z-function for τ if and only if there is a $\mathbb{K}$-homomorphism (resp. $\mathbb{K}$-monomorphism) $\tau' : \mathbb{F}(t) \to S(\mathbb{K})$.*
2. *Let Z and L be z- and o-functions for τ. Then there is a $\mathbb{K}$-homomorphism (resp. $\mathbb{K}$-monomorphism) $\tau' : \mathbb{F}(t) \to S(\mathbb{K})$ with a z-function Z' and an o-function L'. If Z and L are computable, then Z' and L' are computable.*

Proof. (1) The direction from right to left follows by Lemma 4.3. Suppose that Z is a z-function for τ. Let $\frac{p}{q} \in \mathbb{F}(t)$ be in reduced representation. Then we extend ev to $\mathbb{F}(t)$ by

$$\mathrm{ev}(\frac{p}{q}, k) = \begin{cases} 0 & \text{if } k < Z(q) \\ \frac{\mathrm{ev}(p,k)}{\mathrm{ev}(q,k)} & \text{if } k \geq Z(q). \end{cases} \tag{4.8}$$

The properties (4.2), (4.3), (4.4), (4.5) can be carried over from $\mathbb{F}[t]$ to $\mathbb{F}(t)$. By Lemma 4.1 we get a $\mathbb{K}$-homomorphism $\tau' : \mathbb{F}(t) \to S(\mathbb{K})$ with (4.1). Finally, suppose that τ is injective. Take $f = \frac{p}{q}$ in reduced form such that $\mathbf{0} = \tau'(f) = \frac{\tau(p)}{\tau(q)}$. Since $\mathrm{ev}(q, k) \neq 0$ for all $k \geq Z(q)$, $\tau(p) = 0$. As τ is injective, $p = 0$ and thus $f = 0$. This proves that τ' is injective.

(2) Let L and Z be o- and z-functions for τ, respectively. Then we extend them to $\mathbb{F}(t)$ by

$$
\begin{aligned}
Z'(\frac{p}{q}) &= \begin{cases} Z(p) & \text{if } q = 1 \\ \max(Z(p), Z(q)) & \text{if } q \neq 1, \end{cases} \\
L'(\frac{p}{q}) &= \begin{cases} L(p) & \text{if } q = 1 \\ \max(L(p), L(q), Z(q)) & \text{if } q \neq 1 \end{cases}
\end{aligned}
\tag{4.9}
$$

where $\frac{p}{q} \in \mathbb{F}(t)$ is in reduced representation. By construction Z' and L' are z- and o-functions for τ'. If L and Z are computable, then Z' and L' are computable. □

Remark 4.7. Given the $\mathbb{K}$-homomorphism (resp. $\mathbb{K}$-monomorphism) $\tau : \mathbb{F}[t] \to S(\mathbb{K})$, the $\mathbb{K}$-homomorphism (resp. $\mathbb{K}$-monomorphism) $\tau' : \mathbb{F}(t) \to S(\mathbb{K})$ with $\tau'(p) = \tau(p)$ for all $p \in \mathbb{F}[t]$ is uniquely determined by (4.8) – up to the choice of the z-function Z.

Example 4.8. Let $(\mathbb{K}(n), \sigma)$ be the $\Pi\Sigma^*$-field over $\mathbb{K}$ with $\sigma(n) = n + 1$. Then by our construction we get a $\mathbb{K}$-monomorphism $\tau : \mathbb{K}(n) \to S(\mathbb{K})$ with computable o- and z-functions as follows: Start with the $\mathbb{K}$-monomorphism $\tau : \mathbb{K} \to S(\mathbb{K})$ with $\tau(k) = \boldsymbol{k} = (k, k, \dots)$ and take the o-function $L(k) = 0$ and the z-function $Z(k) = 0$ for all $k \in \mathbb{K}$. By Lemma 4.4 we get the $\mathbb{K}$-monomorphism $\tau : \mathbb{K}[n] \to S(\mathbb{K})$ defined by $\text{ev}(p, k) = p(k)$ for all $p \in \mathbb{K}[n]$ and all $k \geq 0$. The resulting o-function is $L(p) = 0$ for all $p \in \mathbb{K}[n]$. Note that the z-function exists since $p(n) \in \mathbb{K}[n]$ can have only finitely many roots. The nonnegative integer roots can be easily computed; see, e.g., [PWZ96, page 80]. Hence by Theorem 4.6 we can lift the $\mathbb{K}$-monomorphism from $\mathbb{K}[n]$ to $\mathbb{K}(n)$ with the o-function L' and z-function Z' given by (4.9).

Lemma 4.9. *Let $(\mathbb{K}(q)(t_1) \dots (t_e)(t), \sigma)$ be a $\Pi\Sigma^*$-field over the rational function field $\mathbb{K}(q)$ where $\sigma(t_i) = \alpha_i t_i + \beta_i$ with $\alpha_i, \beta_i \in \mathbb{K}$ and $\sigma(t) = q\,t$. If there is a $\mathbb{K}$-monomorphism $\tau : \mathbb{K}(q)(t_1) \dots (t_e) \to S(\mathbb{K}(q))$ with (computable) o- and z-functions, there is a $\mathbb{K}$-monomorphism $\tau : \mathbb{K}(q)(t_1) \dots (t_e)(t) \to S(\mathbb{K}(q))$ with (computable) o- and z-functions.*

Proof. By Lemma 4.4 there is a $\mathbb{K}$-monomorphism $\tau' : \mathbb{K}(q)(t_1) \dots (t_e)[t] \to S(\mathbb{K}(q))$ with an o-function L'; L' is computable if L is computable. In this construction we can take $\text{ev}(t, k) = q^k$. By [BP99, Sec. 3.7] there is a Z'-function for $\mathbb{K}(q)(t_1) \dots (t_e)[t]$; it is computable, if Z is computable. By Theorem 4.6 we get a $\mathbb{K}$-monomorphism from $\mathbb{K}(q)(t_1) \dots (t_e)(t)$ to $S(\mathbb{K}(q))$ with o- and z-functions; they are computable, if L', Z' are computable. □

By Expample 4.8 and iterative application of Lemma 4.9 based on [BP99] we get the mixed case.

Corollary 4.10. *Let $(\mathbb{K}(n)(t_1) \dots (t_e), \sigma)$ be a $\Pi\Sigma^*$-field over a rational function field $\mathbb{K} := \mathbb{K}'(q_1) \dots (q_e)$ where $\sigma(n) = n + 1$ and $\sigma(t_i) = q_i t_i$ for all $1 \leq i \leq e$. Then there is a $\mathbb{K}$-monomorphism $\tau : \mathbb{K}(n)(t_1) \dots (t_e) \to S(\mathbb{K})$ with a computable o-function and z-function.*

Note that the use of asymptotic arguments might produce $\mathbb{K}$-monomorphisms for more general $\Pi\Sigma^*$-fields. An open question is, if any $\Pi\Sigma^*$-field over $\mathbb{K}$ can be embedded in $S(\mathbb{K})$.

5. A criterion to check algebraic independence

We consider the following application. Given sums of the type (1.3), we start with an appropriate $\Pi\Sigma^*$-field $(\mathbb{F}, \sigma)$ (e.g., the rational case, q-rational case, or the mixed case) and construct, if possible, a generalized d'Alembertian extension $(\mathbb{F}(t_1) \dots (t_e), \sigma)$ of $(\mathbb{F}, \sigma)$ together with a $\mathbb{K}$-monomorphism such that for each $1 \leq i \leq d$: $f_i \in \mathbb{F}[t_1, \dots, t_e]$ with $\text{ev}(f_i, k) = f_i(k)$

for all $k \geq r$ for some $r \in \mathbb{N}$. Here one must choose within the monomorphism construction the initial values c in (4.6) accordingly.

Remark. In Sigma this translation mechanism is done automatically; see, e.g., Section 8.

Then one can prove or disprove the transcendence of the sums (1.3) by showing the non-existence or existence of a parameterized telescoping solution (3.1) in $\mathbb{F}[t_1, \dots, t_e]$. This fact can be summarized in

Theorem 5.1 (Main result). *Let $(\mathbb{F}(t_1)\dots(t_e), \sigma)$ be a generalized d'Alembertian-extension of $(\mathbb{F}, \sigma)$ with $\mathbb{K} := \text{const}_\sigma \mathbb{F}$, and let $\tau : \mathbb{F}[t_1, \dots, t_e] \to S(\mathbb{K})$ be a $\mathbb{K}$-monomorphism with (4.1) together with an o-function; let $(f_1, \dots, f_d) \in \mathbb{F}[t_1, \dots, t_e]^d$. Then the following statements are equivalent:*

1. *There are no $g \in \mathbb{F}[t_1, \dots, t_e]$ and $\mathbf{0} \neq (c_1, \dots, c_d) \in \mathbb{K}^d$ with (3.1).*
2. *The sequences $\{(S_1(n))_{n\geq 0}, \dots, (S_d(n))_{n\geq 0}\}$ given by*
$$S_1(n) := \sum_{k=r}^{n} \text{ev}(f_1, k), \dots, S_d(n) := \sum_{k=r}^{n} \text{ev}(f_d, k). \tag{5.1}$$
for some r big enough, are algebraically independent over $\tau(\mathbb{F}[t_1, \dots, t_e])$.

Proof. Suppose (1) holds. Then by Theorem 2.7 and Theorem 3.1 there is the Σ^*-extension $(\mathbb{F}(t_1)\dots(t_e)(s_1)\dots(s_d), \sigma)$ of $(\mathbb{F}(t_1)\dots(t_e), \sigma)$ with
$$\sigma(s_1) = s_1 + f_1, \dots, \sigma(s_d) = s_d + f_d.$$
Moreover, there is a $\mathbb{K}$-monomorphism $\tau' : \mathbb{F}[t_1, \dots, t_e][s_1, \dots, s_d] \to S(\mathbb{K})$ defined by $\tau'(f) = \tau(f)$ for all $f \in \mathbb{F}[t_1, \dots, t_e]$ and $\tau'(s_i) = (S_i(n))_{n\geq 0}$ for $1 \leq i \leq d$ where (5.1) for some r big enough. Since $R[s_1, \dots, s_d]$ is a polynomial ring over $R := \mathbb{F}[t_1, \dots, t_e]$, (2) follows by the $\mathbb{K}$-monomorphism τ'.
Conversely, suppose that (1) does not hold. Then we get (1.1) with $g(k) := \text{ev}(g, k)$ and $f_i(k) := \text{ev}(f_i, k)$ with k big enough, say $k \geq r$. Summing this equation over $r \leq k \leq n$ gives a relation of the form (1.2), i.e., the sums in (1.3) are algebraic. Thus (2) does not hold. □

From the algorithmic point of view, Sigma can check the non-existence of a solution of (3.1), which then implies the transcendence of the sums (5.1). Note that an appropriate r in (5.1) is computable if τ has a computable o- and z-function.

Besides this, restricting the f_i to elements with certain structure allows to predict the non-existence of a solution of (3.1). In this way, we are able to classify various types of sums to be algebraically independent. Subsequently, we will explore these aspects for various types of sums.

6. Rational sums

Applying Theorem 5.1 together with Example 4.8 gives the following theorem.

Theorem 6.1. *Let $f_1(k), \dots, f_d(k) \in \mathbb{K}(k)$. If there are no $g(k) \in \mathbb{K}(k)$ and $c_1, \dots, c_d \in \mathbb{K}$ with (1.1) then the sequences (1.3), for some r big enough, are algebraically independent over $\mathbb{K}(n)$, i.e., there is no polynomial $P(x_1, \dots, x_d) \in \mathbb{K}(n)[x_1, \dots, x_d]^*$ with*
$$P(S_1(n), \dots, S_d(n)) = 0 \quad \forall n \geq 0. \tag{6.1}$$

Corollary 6.2. *Let $p_1(k), p_2(k), \dots \in \mathbb{K}[k]^*$, $u_1(k), u_2(k), \dots \in \mathbb{K}[k]^*$ and $q \in \mathbb{K}[k]^*$ with $\deg(q) > 0$ and $\gcd(p_i, q) = \gcd(u_i, q) = 1$ for all $i \geq 1$; suppose that $q(r) \neq 0$ for all $r \in \mathbb{N}^*$ and $\gcd(q(k), q(k+r)) = 1$ for all $r \in \mathbb{N}^*$. Then the sums*

$$S_1(n) := \sum_{k=1}^{n} u_1(k) \left(\frac{p_1(k)}{q(k)}\right), S_2(n) := \sum_{k=1}^{n} u_2(k) \left(\frac{p_2(k)}{q(k)}\right)^2, \dots$$

are algebraically independent over $\mathbb{K}(n)$, i.e., there is no $P(x_1, \dots, x_d) \in \mathbb{K}(n)[x_1, \dots, x_d]^*$ for some $d \geq 1$ with (6.1).

Proof. Denote $f_i(k) := u_i\left(\frac{p_i}{q}\right)^i$ and suppose there are $g(k) \in \mathbb{K}(k)$ and $c_i \in \mathbb{K}$ with (1.1) where $d \geq 1$ is minimal. Then it follows that

$$g(k+1) - g(k) = \frac{c_1 u_1 p_1 q^{d-1} + c_2 u_2 p_2^2 q^{d-2} + \dots + c_d u_d p_d^d}{q^d} =: \frac{v}{q^d}.$$

Since $c_d \neq 0$, $\gcd(q, c_d u_d p_d^d) = 1$. Hence $\gcd(v, q) = 1$, and thus $\gcd(v, q^d) = 1$. By Theorem 2.8, $g(k) \in \mathbb{K}(k)$ cannot exist; a contradiction. The corollary follows by Theorem 6.1. $\square$

Example 6.3. Choosing $p_i = u_i = 1, q = k$ in Corollary 6.2 proves that the generalized harmonic numbers $\{H_n^{(i)} | i \geq 1\}$ are algebraically independent over $\mathbb{K}(n)$.

Applying Theorem 5.1 together with Corollary 4.10 accordingly produces the q-versions and the mixed versions of Theorem 6.1 and Corollary 6.2. A typical application is Example 6.4.

Example 6.4. The q-harmonic numbers $\{\sum_{k=1}^{n} \frac{1}{(1-q^k)^i} | i \geq 1\}$ (or for instance the variations $\{\sum_{k=1}^{n} \left(\frac{q^k}{1-q^k}\right)^i | i \geq 1\}$) are algebraically independent over $\mathbb{K}(q^k)$.

Completely analogously to Corollary 6.2 one can show the following corollary

Corollary 6.5. *Let $p_1(k), q_1(k), \dots, p_d(k), q_d(k) \in \mathbb{K}[k]^*$ with $\deg(q_i) > 0$ and $\gcd(p_i, q_i) = 1$. Suppose that $q_i(k) \neq 0$ for all $k \in \mathbb{N}$ and that $\gcd(q_i(k+r), q_j(k)) = 1$ for all $r \in \mathbb{Z}$ and all $1 \leq i < j \leq d$. Then the sums*

$$S_1(n) := \sum_{k=1}^{n} \frac{p_1(k)}{q_1(k)}, \dots, S_d(n) := \sum_{k=1}^{n} \frac{p_d(k)}{q_d(k)}$$

are algebraically independent over $\mathbb{K}(n)$, i.e., there is no $P(x_1, \dots, x_d) \in \mathbb{K}(n)[x_1, \dots, x_d]^*$ with (6.1).

7. Hypergeometric sums and the minimality of recurrences

Suppose that $f(k)$ is a hypergeometric term in k, i.e., there is an $\alpha \in \mathbb{K}(k)$ with $\alpha(r) := \frac{f(r+1)}{f(r)}$ for all r big enough; in short we also write $\alpha(k) := \frac{f(k+1)}{f(k)}$ to define the rational function $\alpha \in \mathbb{K}(k)$. In this context the following result is important. $f(k)$ can be represented with t in the $\Pi\Sigma^*$-field $(\mathbb{K}(k)(t), \sigma)$ over $\mathbb{K}$ with $\sigma(k) = k+1$ and $\sigma(t) = \alpha\, t$ if and only if there are no $r(k) \in \mathbb{K}(k)$ and no root of unity γ with $f(k) = \gamma^k r(k)$; see [Sch05b, Thm. 5.4]. In the following, we exclude this special case.

Subsequently, we exploit Corollary 2.9 for the hypergeometric case.

Corollary 7.1. *Let $f(k)$ be a hypergeometric term such that $f(k) \neq \gamma^k r(k)$ for all $r(k) \in \mathbb{K}(k)$ and all roots of unity γ, and consider the $\Pi\Sigma^*$-field $(\mathbb{K}(k)(t), \sigma)$ over $\mathbb{K}$ with $\sigma(k) = k+1$ and $\sigma(t) = \frac{f(k+1)}{f(k)}t$. Then $\sigma(g) - g = w\,t$, if and only if $g = v\,t + c$ for $v \in \mathbb{K}(k)$, $c \in \mathbb{K}$ where*

$$\alpha(k)\,v(k+1) - v(k) = w. \tag{7.1}$$

We note that (7.1) is nothing else than the basic ansatz [PWZ96, Equ. 5.2.2] of Gosper's algorithm. In a nutshell, Gosper's algorithm (and also Zeilberger's algorithm) check the existence of a solution in the corresponding $\Pi\Sigma^*$-field $(\mathbb{K}(k)(t), \sigma)$.

As a consequence, Theorem 5.1 can be simplified to the following version.

Theorem 7.2. *Let $f(k)$ be a hypergeometric term such that $f(k) \neq \gamma^k r(k)$ for all $r(k) \in \mathbb{K}(k)$ and all roots of unity γ, and consider the $\Pi\Sigma^*$-field $(\mathbb{K}(k)(t), \sigma)$ over $\mathbb{K}$ with $\sigma(k) = k+1$ and $\sigma(t) = \frac{f(k+1)}{f(k)}t$. Let $r_i(k) \in \mathbb{K}(k)$ for $1 \leq i \leq d$ and set $f_i := r_i t \in \mathbb{K}(k)(t)$. If there are no $c_i \in \mathbb{K}$ and $w \in \mathbb{K}(k)$ with $g := w\,t$ such that (3.1), then the following sequences, for r big enough, are algebraically independent over $\mathbb{K}(n)$:*

$$f(n),\ S_1(n) = \sum_{k=r}^{n} r_1(k) f(k), \dots, S_d(n) = \sum_{k=r}^{b} r_d(k) f(k),$$

i.e., there is no $P(x_0, x_1, \dots, x_d) \in \mathbb{K}(n)[x_0, x_1, \dots, x_d]^*$ with

$$P(f(n), S_1(n), \dots, S_d(n)) = 0 \quad \forall n \geq 0. \tag{7.2}$$

Proof. Suppose there is a solution $c_i \in \mathbb{K}$ and $g \in \mathbb{K}(k)(t)$ with (3.1). By Corollary 2.9, $\sigma(wt) - wt = t\sum_{i=1}^{d} c_i r_i = \sum_{i=1}^{d} c_i\, f_i$ for some $w \in \mathbb{F}$; a contradiction to the assumption. Applying Theorem 5.1 and choosing an appropriate $\mathbb{K}$-monomorphism proves the theorem. $\square$

In particular, in the context of finding recurrences we obtain the following result.

Corollary 7.3. *Let $f(\boldsymbol{m}, k)$ be a hypergeometric term in $\boldsymbol{m} = (m_1, \dots, m_u)$ and in k such that $f(\boldsymbol{m}, k) \neq \gamma^k r(\boldsymbol{m}, k)$ for all $r(\boldsymbol{m}, k) \in \mathbb{K}(\boldsymbol{m}, k)$ and all roots of unity γ. Let $S = \{s_1, \dots, s_d\} \subseteq \mathbb{Z}^d$. Consider the $\Pi\Sigma^*$-field $(\mathbb{K}(\boldsymbol{m})(k)(t), \sigma)$ over $\mathbb{K}(\boldsymbol{m})$ with $\sigma(k) = k+1$ and $\sigma(t) = \frac{f(\boldsymbol{m},k+1)}{f(\boldsymbol{m},k)}t$, and define $f_i := \frac{f(\boldsymbol{m}+\boldsymbol{s_i},k)}{f(\boldsymbol{m},k)}t \in \mathbb{K}(\boldsymbol{m})(k)(t)$. If there are no $c_i \in \mathbb{K}(\boldsymbol{m})$ and $w \in \mathbb{K}(\boldsymbol{m})(k)$ such that (3.1) for $g := w\,t$, then the following sequences, for r big enough, are algebraically independent over[3] $\mathbb{K}(\boldsymbol{m})(n)$:*

$$S_0(n) = f(\boldsymbol{m}, n),\ S_1(n) = \sum_{k=r}^{n} f(\boldsymbol{m} + \boldsymbol{s_1}, k), \dots, S_d(n) = \sum_{k=r}^{n} f(\boldsymbol{m} + \boldsymbol{s_d}, k),$$

i.e., there is no $P(x_0, x_1, \dots, x_d) \in \mathbb{K}(\boldsymbol{m})(n)[x_0, x_1, \dots, x_d]^*$ with (7.2).

Moreover, if one applies Zeilbergers's creative telescoping algorithm [Zei91], the result can be reduced to the following

Corollary 7.4. *Let $f(m, k)$ be a hypergeometric term in m and in k such that $f(m, k) \neq \gamma^k r(m, k)$ for all $r(m, k) \in \mathbb{K}(m, k)$ and all roots of unity γ. If Zeilberger's algorithm fails to find $c_i(m) \in \mathbb{K}(m)$ and $g(m, k)$ such that*

$$g(m, k+1) - g(m, k) = c_0(m) f(m, k) + \dots + c_d(m) f(m+d, k),$$

[3] Here $\mathbb{K}(\boldsymbol{m}) = \mathbb{K}(m_1, \dots, m_u)$ is a rational function field.

then the sequence $S_0(n) = f(m,n)$ *in* n *and the sums (for some* r *big enough)*

$$S_1(n) = \sum_{k=r}^{n} f(m,k), \dots, S_d(n) = \sum_{k=r}^{n} f(m+d,k) \tag{7.3}$$

are algebraically independent over $\mathbb{K}(m)(n)$*, i.e., there is no polynomial* $P(x_0, x_1, \dots, x_d) \in \mathbb{K}(m)(n)[x_0, x_1, \dots, x_d]^*$ *with* (7.2).

As a consequence, Zeilberger's algorithm finds a recurrence with minimal order for sums of the type (7.3). Even more, it shows algebraic independence of the sums!

Example 7.5. For the Apéry-sum $S(m) = \sum_{k=0}^{m} \binom{m}{k}^2 \binom{m+k}{k}$, see [Poo79], Zeilberger's algorithm finds a recurrence of order 2, but not smaller ones. Hence, the following sequences in n are algebraically independent over $\mathbb{K}(m)(n)$:

$$\binom{m}{n}^2 \binom{m+n}{n}, \sum_{k=0}^{n} \binom{m}{k}^2 \binom{m+k}{k}, \sum_{k=0}^{n} \binom{m+1}{k}^2 \binom{m+k+1}{k}.$$

The following remarks are in place.

(1) We consider m as an indeterminate; if m is replaced by specific integers, the sums in (7.3) might be not well defined because of poles. In particular, r might be chosen too small, or n cannot be arbitrarily large.

(2) Moreover, the situation might drastically change, if we consider, e.g., sums of the type

$$S_1(m) = \sum_{k=r}^{am+b} f(m,k), \dots, S_d(m) = \sum_{k=r}^{a(m+d)+b} f(m+d,k), \tag{7.4}$$

for integers a, b. In this case, the minimal order of the corresponding recurrence might be lower.

Example 7.6. Consider the sum

$$S_d(m,n) = \sum_{k=0}^{n} (-1)^k \binom{m}{k} \binom{d\,k}{m}$$

for integers $d \geq 1$. As worked out in [PS95, Sec. 4.3], Zeilberger's algorithm finds only a recurrence of order $o_d = \max(d-1, 1)$. Hence the sequence $f(n) = (-1)^n \binom{m}{n} \binom{d\,n}{m}$ and the sums

$$S_d(m,n), \dots, S_d(m + o_d - 1, n)$$

in n are algebraically independent over $\mathbb{Q}(m)(n)$. But, if we set $n = m$, the situation changes drastically. In this particular case,

$$S_d(n,n) = (-d)^n,$$

in other words, only the sequences $f(n)$ and $S_d(n,n)$, $d > 1$, are algebraically independent over $\mathbb{Q}$.

To sum up, Zeilberger's algorithm finds, in case of existence, a recurrence with minimal order for sums of the type (7.3). And it does not succeed in this task, if the specialization to (7.4) introduces additional linear recurrence relations with smaller order.

In [Abr03] a criterion is given when Zeilberger's algorithm fails to find a creative telescoping solution for a hypergeometric input summand $f(m,k)$. If $f(m,k)$ satisfies this criterion, then all the sequences $h(m,n), S(m,n), S(m+1,n), \dots$ are algebraically independent over $\mathbb{K}(m)$.

Example 7.7. Since

$$f(m,k)=\frac{1}{mk+1}(-1)^k\binom{m+1}{k}\binom{2m-2k-1}{m-1}$$

satisfies Abramov's criterion, see [Abr03, Exp. 2], it follows that the sequences $f(m,n)$ and $\{S(m+i,n)|i\geq 0\}$ in n with $S(m,n)=\sum_{k=0}^{n} f(m,k)$ are algebraically independent over $\mathbb{Q}(m)(n)$.

Another criterion for transcendence is the following result inspired by [PWZ96, Sec. 5.6].

Theorem 7.8. *Let $f_1(k),\dots,f_d(k)$ be hypergeometric terms with the following properties:*

(1) *There is a $\Pi\Sigma^*$-field $(\mathbb{K}(k)(t_1)\dots(t_d),\sigma)$ over $\mathbb{K}$ with $\sigma(k)=k+1$ and with $\sigma(t_i)=\alpha_i t_i$ where $\alpha_i:=\frac{f_i(k+1)}{f_i(k)}\in\mathbb{K}(k)$ for all $1\leq i\leq d$.*

(2) *For all $1\leq i\leq d$, $f_i(k)$ is not Gosper-summable, i.e., $\nexists g(k)\in\mathbb{K}(k)$ with $\alpha_i\,g(k+1)-g(k)=1$.*

Then the sequences $f_1(n),\dots,f_d(n)$ together with $S_1(n),\dots,S_d(n)$ from (1.3), r big enough, are algebraically independent over $\mathbb{K}(n)$, i.e., there is no $P(x_1,\dots,x_{2d})\in\mathbb{K}(n)[x_1,\dots,x_{2d}]^*$ with

$$P(f_1(n),\dots,f_d(n),S_1(n),\dots,S_d(n))=0\quad\forall n\geq 0.$$

Proof. Denote $\mathbb{F}:=\mathbb{K}(k)(t_1)\dots(t_d)$ and suppose that there are $\mathbf{0}\neq(c_1,\dots,c_d)\in\mathbb{K}^d$ and $g\in\mathbb{K}(k)[t_1,\dots,t_e]$ with (3.1) where $f_i:=t_i$. By [Kar81, Cor. 2] or [Sch05a, Cor. 3], $g=\sum_{i=1}^d w_i t_i+u$ with $w_i,u\in\mathbb{K}(k)$. Plugging g into (3.1) and doing coefficient comparison (the t_i are transcendental!) shows that $\sigma(w_i t_i)-w_i t_i=c_i t_i$ for all $1\leq i\leq d$. By property (2), $c_i=0$ for all i; a contradiction to the assumption. Applying Theorem 5.1 and choosing an appropriate $\mathbb{K}$-monomorphism proves the theorem. $\square$

Example 7.9. The sequences $\{n!,\binom{m}{n},(n+m)!,\sum_{k=1}^n k!,\sum_{k=1}^n\binom{m}{k},\sum_{k=1}^n(k+m)!\}$ in n are algebraically independent over $\mathbb{K}(m)(n)$.

We note that the q-hypergeometric case can be handled completely analogously with our machinery.

8. Nested sums

Most of the ideas of Section 7 can be carried over to sequences in terms of generalized d'Alembertian extensions. E.g., in [PS03] we derived for the sum

$$S(m):=\sum_{k=0}^{m}(1+5\,H_k(m-2k))\binom{m}{k}^5$$

a recurrence of order 4 with creative telescoping, but failed to find a recurrence of smaller order. Hence Theorem 5.1 tells us that the sequences

$$\Big(\binom{m}{n}\Big)_{n\geq 0},\ \big(H_n\big)_{n\geq 0},\ \big(S(m,n)\big)_{n\geq 0},\dots,\ \big(S(m+3,n)\big)_{n\geq 0}\tag{8.1}$$

in n with

$$S(m,n):=\sum_{k=0}^{n}f(m,k)=\sum_{k=0}^{n}(1+5\,H_k(m-2k))\binom{m}{k}^5$$

are algebraically independent over $\mathbb{K}(m)(n)$. Internally, Sigma works as follows: It constructs the $\Pi\Sigma^*$-field $(\mathbb{F},\sigma)$ with $\mathbb{F} := \mathbb{Q}(m)(k)(b)(h)$ from Example 2.2 and designs the $\mathbb{Q}(m)$-monomorphism with

$$\mathrm{ev}(k,j)=j, \qquad \mathrm{ev}(b,j)=\prod_{i=1}^{j}\frac{m+1-i}{i}=\binom{m}{j}, \qquad \mathrm{ev}(h,j)=\sum_{i=1}^{j}\frac{1}{k}=H_j.$$

Then it takes

$$f_1 = b^5(1+5h(m-2k)), \qquad f_2=\tfrac{b^5(m+1)^5(5h(-2k+m+1)+1)}{(-k+m+1)^5},$$
$$f_3=\tfrac{b^5(m+1)^5(m+2)^5(5h(-2k+m+2)+1)}{(-k+m+1)^5(-k+m+2)^5}, \qquad f_4=\tfrac{b^5(m+1)^5(m+2)^5(m+3)^5(5h(-2k+m+3)+1)}{(-k+m+1)^5(-k+m+2)^5(-k+m+3)^5};$$

this is motivated by the fact $\binom{m+1}{k}=\frac{m+1}{m+1-k}\binom{m}{k}$ which shows that $\mathrm{ev}(f_i,k)=f(m+i-1,k)$. Finally, Sigma proves algorithmically that there are no $g\in\mathbb{F}$ and $c_i\in\mathbb{Q}(m)$ with (3.1). Hence the transcendence of (8.1) follows by Theorem 5.1.

We note that the sum $S(n)=S(n,n)$ has completely different properties: it satisfies a recurrence of order two. More precisely, as shown in [PS03] we get

$$\sum_{k=0}^{n}(1+5H_k(n-2k))\binom{n}{k}^5=(-1)^n\sum_{k=0}^{n}\binom{n}{k}^2\binom{n+k}{j}.$$

9. A transcendence criterion for products

The product version of Theorem 3.1 is Theorem 9.1.

Theorem 9.1. *Let $(\mathbb{F},\sigma)$ be a difference field with constant field $\mathbb{K}$ and $(f_1,\dots,f_d)\in(\mathbb{F}^*)^d$. The following statements are equivalent.*

1. *There do not exist $\mathbf{0}\neq(c_1,\dots,c_d)\in\mathbb{Z}^d$ and $g\in\mathbb{F}^*$ with*
$$\frac{\sigma(g)}{g}=f_1^{c_1}\dots f_d^{c_d}. \tag{9.1}$$
2. *There is a Π-extension $(\mathbb{F}(t_1,\dots,t_d),\sigma)$ of $(\mathbb{F},\sigma)$ with $\sigma(t_i)=f_it_i$ for $1\leq i\leq d$.*

Proof. Suppose that $(\mathbb{F}(t_1,\dots,t_d),\sigma)$ is a Π-extension of $(\mathbb{F},\sigma)$. Moreover, assume that there is a $g\in\mathbb{F}^*$ and $\mathbf{0}\neq(c_1,\dots,c_d)\in\mathbb{Z}^d$ with (9.1). Let j be maximal with $c_j\neq 0$. Then with $w=gt_1^{-c_1}\dots t_{j-1}^{-c_{j-1}}\in\mathbb{F}(t_1,\dots,t_{j-1})^*$ we get $\sigma(w)=f_j^{c_j}w$, a contradiction to Theorem 2.3.2. Conversely, let j be maximal such that $(\mathbb{F}(t_1,\dots,t_j),\sigma)$ is a Π-extension of $(\mathbb{F},\sigma)$; suppose that $j<d$. By Theorem 2.3.2 there is a $g\in\mathbb{F}(t_1,\dots,t_{j-1})^*$ and $c\in\mathbb{Z}$ with $\sigma(g)=f_j^cg$. By Theorem 2.4.2 it follows that $g=u\,t_1^{c_1}\dots t_{j-1}^{c_{j-1}}$ with $c_i\in\mathbb{Z}$ and $u\in\mathbb{F}$; clearly $u\neq 0$. Thus, $\frac{\sigma(u)}{u}=f_1^{-c_1}\dots f_{j-1}^{-c_{j-1}}f_j^c$ which proves the theorem. $\square$

Note that the existence of a solution of (9.1) can be checked by Karr's algorithm [Kar81] if $(\mathbb{F},\sigma)$ is a $\Pi\Sigma^*$-field over $\mathbb{K}$. The following theorem is immediate.

Theorem 9.2. *Let $(\mathbb{F},\sigma)$ be a difference field with constant field $\mathbb{K}$, let $\tau:\mathbb{F}\to S(\mathbb{K})$ be a $\mathbb{K}$-monomorphism together with an o-function, and let $(f_1,\dots,f_d)\in(\mathbb{F}^*)^d$. Then the following statements are equivalent:*

1. *There are no $g\in\mathbb{F}^*$ and $\mathbf{0}\neq(c_1,\dots,c_d)\in\mathbb{Z}^d$ with (9.1).*

(2) *The sequences* $(S_1(n))_{n\geq 0},\dots,(S_d(n))_{n\geq 0}$ *given by*

$$S_1(n):=\prod_{k=r}^{n}\mathrm{ev}(f_1,k),\dots,S_d(n):=\prod_{k=r}^{n}\mathrm{ev}(f_d,k),$$

for some r *big enough, are algebraically independent over* $\tau(\mathbb{F})$.

Example 9.3. The sequences $2^n,3^n,5^n,7^n,\dots$ over the prime numbers are algebraically independent over $\mathbb{K}(n)$; compare [Kar81, Exp. 7].

The following lemma is a direct consequence of [Sch05b, Thm. 4.14]; for the rational case see also [AP02].

Lemma 9.4. *Let* $(\mathbb{F}(t),\sigma)$ *be a* $\Pi\Sigma^*$*-extension of* $(\mathbb{F},\sigma)$*. Let* $p_1,\dots,p_d,q_1,\dots,q_d\in\mathbb{F}[t]^*$ *such that* $\gcd(\sigma^l(p_i),q_j)=1$ *for all* $l\in\mathbb{Z}$ *and* $1\leq i<j\leq d$*; set* $f_i:=\frac{p_i}{q_i}$*. Then there is no* $g\in\mathbb{F}(t)^*$ *and* $(c_1,\dots,c_d)\in\mathbb{Z}^d$ *with* (9.1).

Corollary 9.5. *Let* $p_1(k),q_1(k)\dots,p_d(k),q_d(k)\in\mathbb{K}[k]$ *with* $\gcd(p_i(k+l),q_j(k))=1$ *for all* i,j *and* $l\in\mathbb{Z}$*. Then the sequences*

$$S_1(n):=\prod_{k=r}^{n}\frac{p_1(k)}{q_1(k)},\dots,S_d(n):=\prod_{k=r}^{n}\frac{p_d(k)}{q_d(k)},$$

for some r *big enough, are algebraically independent over* $\mathbb{K}(n)$*, i.e., there is no polynomial* $P(x_1,\dots,x_d)\in\mathbb{K}(n)[x_1,\dots,x_d]^*$ *with* (6.1).

10. Conclusion

We showed that telescoping, creative telescoping and, more generally, parameterized telescoping can be applied to obtain a criterion to check algebraic independence of nested sum expressions. For sums over hypergeometric terms any implementation of Zeilberger's algorithm can be used to check transcendence. In general, the summation package Sigma can be applied to check algebraic independence of indefinite nested sums and products.

Moreover, using results from summation theory one can show that whole classes of sums are transcendental. Obviously, refinements of summation theory should give also stronger tools to prove or disprove transcendence of sum expressions. E.g., Peter Paule's results [Pau04] enable one to predict the existence of contiguous relations. Using these results might help to refine, e.g., Corollary 7.3.

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